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Local cohomology and Serre subcategories

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Abstract

The membership of the local cohomology modules $\mathrm{H}^n_{\mathfrak{a}}(M)$ of a module M in certain Serre subcategories of the category of modules is studied from below ($i < n$) and from above ($i > n$). Generalizations of depth and regular sequences are defined. The relation of these notions to local cohomology are found. It is shown that the membership of the local cohomology modules of a finite module in a Serre subcategory in the upper range just depends on the support of the module.

Keywords: Local cohomology; Serre subcategory; $\mathcal{S}$-regular sequences; $\mathcal{S}$-depth

1. Introduction

Throughout R is a commutative noetherian ring. For unexplained terminology from homological and commutative algebra we refer to [6] and [16]. In Section 2 we will investigate the following question:

Question. Let there be given a Serre subcategory $\mathcal{S}$ of the category of R-modules, an ideal $\mathfrak{a}$ and an R-module M. When do the local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ belong to $\mathcal{S}$ for all $i < n$?

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We will give the answer when $\mathcal{S}$ is closed under taking injective hulls, or more generally when $\mathcal{S}$ is required to satisfy a certain condition, which we call $C_\mathfrak{a}$. This condition says that M is in $\mathcal{S}$, provided that $0:_M \mathfrak{a}$ is in $\mathcal{S}$ and $\operatorname{Supp}_R(M) \subset \mathrm{V}(\mathfrak{a})$. This condition is suitable in induction arguments (cf. [5, Theorem 7.1.2]). The answer to the question above is given in 2.9, in terms of Ext-modules, Koszul cohomology, generalized local cohomology and when M is finite (i.e. finitely generated) in terms of $\mathcal{S}$-sequences. $\mathcal{S}$-sequences are defined in 2.6 and their main properties are formulated in 2.7. When $\mathcal{S}$ satisfies the condition $C_\mathfrak{a}$, we are able to show that if M is a finite module, such that $M/\mathfrak{a}M$ is not in $\mathcal{S}$, all maximal sequences in $\mathfrak{a}$, $\mathcal{S}$-regular on M do have the same length. We call this number $\mathcal{S}\text{-}\operatorname{depth}_\mathfrak{a}(M)$, (see 2.14 and 2.18). Depending on which $\mathcal{S}$ we choose, we obtain various sequences studied in the literature such as regular sequences, filter-regular sequences, generalized regular sequences, etc.

Generalized local cohomology was introduced by Grothendieck [12]. They are defined as the right derived functors of the left exact functor $\Gamma_\mathfrak{a}(\operatorname{Hom}_R(N,-))$. Here N is a finite R-module. They can also be computed as

$$\mathrm{H}^i_\mathfrak{a}(N,M) \cong \varinjlim_{\nu} \operatorname{Ext}^i_R\big(N/\mathfrak{a}^\nu N, M\big)$$

where M is an arbitrary R-module. There are natural isomorphisms $\mathrm{H}^i_\mathfrak{a}(N,M) \cong \operatorname{Ext}^i_R(N,M)$ for all i, when M has support in $\mathrm{V}(\mathfrak{a})$ i.e. when $M = \Gamma_\mathfrak{a}(M)$.

In Section 3 we will study the question when $\mathrm{H}^i_\mathfrak{a}(M)$ is in $\mathcal{S}$ for all $i > n$. Here we consider only finite R-modules M. The answer (for a general $\mathcal{S}$) 3.1, shows that it depends just on the closed subset $\operatorname{Supp}_R(M)$ of $\operatorname{Spec}(R)$. When $\mathcal{S}$ is required to satisfy the condition $C_\mathfrak{a}$, a weaker condition suffices, as we show in 3.3. The results of 3.1 and 3.3 can be reformulated in terms of a generalization of cohomological dimension, 3.6. Cohomological dimension has been studied by Hartshorne [13].

2. A study of local cohomology from below

Recall that a class $\mathcal{S}$ of R-modules is a Serre subcategory of the category of R-modules, when it is closed under taking submodules, quotients and extensions. Let $\mathfrak{a}$ be a fixed ideal of R and let M be an R-module. We will consider a useful condition to be posed on a Serre subcategory of the category of R-modules.

Definition 2.1. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. We say that $\mathcal{S}$ satisfies the condition:

$$(C_\mathfrak{a}) \quad \text{If } M = \Gamma_\mathfrak{a}(M) \quad \text{and} \quad \text{if } 0:_M \mathfrak{a} \text{ is in } \mathcal{S} \quad \text{then } M \text{ is in } \mathcal{S}.$$

Lemma 2.2. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. If $\mathcal{S}$ is closed under taking injective hulls then $\mathcal{S}$ satisfies the condition $C_\mathfrak{a}$.*

Proof. When $M = \Gamma_\mathfrak{a}(M)$, the modules $0:_M \mathfrak{a}$ and M have the same injective hull. □

Lemma 2.3. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. $\mathcal{S}$ satisfies the condition $C_\mathfrak{a}$ if and only if $M = \Gamma_\mathfrak{a}(M)$ and $0:_M x$ is in $\mathcal{S}$ for some $x \in \mathfrak{a}$, implies that M is in $\mathcal{S}$.*

Proof. To prove the nontrivial direction, assume that M is a module with $M = \Gamma_{\mathfrak{a}}(M)$ and that $0 :_M \mathfrak{a}$ is in $\mathcal{S}$.

Put $M_s = 0 :_M (x_1, \dots, x_s)$ for $0 \leqslant s \leqslant r$, where $\mathfrak{a} = (x_1, \dots, x_r)$. By assumption M_r is in $\mathcal{S}$ and our aim is to show that M, which equals M_0, belongs to $\mathcal{S}$. Just observe that for each s with $1 \leqslant s \leqslant r$, we have $0 :_{M_{s-1}} x_s = M_s$ and then use backwards induction on s. □

Example 2.4. The following classes of modules are Serre subcategories closed under taking injective hulls and hence satisfy the condition $C_{\mathfrak{a}}$ by 2.2.

(a) The class of zero modules.
(b) The class of artinian R-modules.
(c) The class of R-modules with finite support.
(d) The class of all R-modules M with $\dim_R M \leqslant s$, where s is a non-negative integer. An R-module M is said to have finite Krull dimension if there is a non-negative integer s, such that the length r of each chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ satisfies $r \leqslant s$ and in this case its dimension, $\dim_R M$, is defined as the least such s. When $s = 0$, we get the class of *semiartinian modules*, i.e. the modules M with $\mathrm{Supp}_R(M) \subset \mathrm{Max}\, R$.
(e) Let $Z \subset \mathrm{Spec}(R)$ be closed under *specialization*, that is if $\mathfrak{q} \supset \mathfrak{p} \in Z$, then $\mathfrak{q} \in Z$.
The class of all R-modules M with $\mathrm{Ass}_R(M) \subset Z$ (equivalently $\mathrm{Supp}_R(M) \subset Z$). For example Z could be a closed set $Z = V(\mathfrak{c})$, for a given ideal $\mathfrak{c}$ of R. If we take $Z = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} \leqslant s\}$ where s is a non-negative integer, then we recover (d).

Example 2.5. For a fixed ideal $\mathfrak{a}$, the category of $\mathfrak{a}$-cofinite artinian modules satisfies the condition $C_{\mathfrak{a}}$ by [18, Proposition 4.1] and 2.3.

However this category is in general not closed under taking injective hulls. When $\mathfrak{a} = (0)$, an artinian module is $\mathfrak{a}$-cofinite if and only if it has finite length. The class of modules having finite length is not closed under taking injective hulls unless R is artinian.

A natural question arises.

Question. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. If $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$ for all ideals $\mathfrak{a}$ of R, is $\mathcal{S}$ closed under taking injective hulls?

Definition 2.6. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules and let M be a module over the noetherian ring R. An element x of R is called $\mathcal{S}$-regular on M if the module $0 :_M x$ is in $\mathcal{S}$.

A sequence $x_1, \dots, x_n$ is an $\mathcal{S}$-sequence on M if x_j is $\mathcal{S}$-regular on $M/(x_1, \dots, x_{j-1})M$ for $j = 1, \dots, n$.

Proposition 2.7. *Let $x_1, \dots, x_n$ be a sequence of elements in R and let M be an R-module.*

(a) *Let $1 \leqslant s \leqslant n$. The sequence $x_1, \dots, x_n$ is an $\mathcal{S}$-sequence on M if and only if $x_1, \dots, x_{s-1}$ is an $\mathcal{S}$-sequence on M and $x_s, \dots, x_n$ is an $\mathcal{S}$-sequence on $M/(x_1, \dots, x_{s-1})M$.*
(b) *Let L be a submodule of M such that $L \in \mathcal{S}$. Then the sequence $x_1, \dots, x_n$ is an $\mathcal{S}$-sequence on M if and only if it is an $\mathcal{S}$-sequence on M/L.*
(c) *Let x, y be an $\mathcal{S}$-sequence on M, then x is $\mathcal{S}$-regular on M/yM.*

Proof. (a) Put $M_s = M/(x_1, \dots, x_{s-1})M$. Then (a) follows directly from the definition of $\mathcal{S}$-regularity, if we note that

$$M_s/(x_s, \dots, x_{j-1})M_s \cong M/(x_1, \dots, x_{j-1})M$$

for $s \leqslant j \leqslant n$.

(b) Put $N = M/L$ and consider the exact sequence

$$0 \to 0:_L x_1 \to 0:_M x_1 \to 0:_N x_1 \to L/x_1L \to M/x_1M \to N/x_1N \to 0.$$

It follows that $0:_M x_1$ is in $\mathcal{S}$, if and only if $0:_N x_1$ is in $\mathcal{S}$. Using induction we get from (a) that $x_2, \dots, x_n$ is $\mathcal{S}$-regular on M/x_1M if and only if $x_2, \dots, x_n$ is $\mathcal{S}$-regular on N/x_1N.

(c) Consider the map $f = x1_M$. We want to apply [18, Lemma 3.1] with $S(N) = 0:_N y$ and $T(N) = N/yN$ for any R-module N. $\operatorname{Ker} f = 0:_M x$ is in $\mathcal{S}$. Hence $T(\operatorname{Ker} f) = \operatorname{Ker} f/y \operatorname{Ker} f$ is in $\mathcal{S}$. $S(\operatorname{Coker} f) = 0:_{M/xM} y$ is also in $\mathcal{S}$. Hence $\operatorname{Ker} T(f)$ is in $\mathcal{S}$ i.e. $0:_{M/yM} x$ is in $\mathcal{S}$ and x is $\mathcal{S}$-regular on M/yM. $\square$

Example 2.8. Let M be an arbitrary module, with $\mathcal{S}$ as in Example 2.4. $\mathcal{S}$-sequences on M are just:

(a) Poor M-sequences, [5, Definition 6.2.1]. (When we also require that $M/(x_1, \dots, x_n)M \neq 0$ we get the ordinary notion of an M-regular sequence.)
(b) Filter-regular sequences which have been defined in the local case for finite modules in [20].
(c) Generalized regular sequences which have been defined in the local case for finite modules in [19, Definition 2.1].
(d) M-sequences in dimension $> s$ which have been defined for finite modules in [4, Definition 2.1].
(e) When $Z = \mathrm{V}(\mathfrak{c})$ for a given ideal $\mathfrak{c}$ of R, they are called $\mathfrak{c}$-filter regular sequences. For a local ring $(R, \mathfrak{m})$ and $\mathfrak{c} = \mathfrak{m}$, we recover (b) in the local case when M is finite.

The following theorem that is one of our main results of this paper, yields a characterization of local cohomology modules that are in a Serre subcategory of the category of R-modules satisfying the condition $C_\mathfrak{a}$ for an ideal $\mathfrak{a}$ of R.

Theorem 2.9. *Let $\mathfrak{a} = (x_1, \dots, x_r)$ be an ideal of a noetherian ring R. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition $C_\mathfrak{a}$. Then for each R-module M and each positive integer n the following conditions are equivalent*:

(i) $\mathrm{H}^i_\mathfrak{a}(M)$ *is in $\mathcal{S}$ for all $i < n$ (for all i).*
(ii) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is in $\mathcal{S}$ for all $i < n$ (for all i).*
(iii) $\operatorname{Ext}^i_R(N, M)$ *is in $\mathcal{S}$ for all $i < n$ and each finite module N such that* $\operatorname{Supp}_R(N) \subset \mathrm{V}(\mathfrak{a})$ *(for all i).*
(iv) *There is a finite R-module N with* $\operatorname{Supp}_R(N) = \mathrm{V}(\mathfrak{a})$ *such that* $\operatorname{Ext}^i_R(N, M)$ *is in $\mathcal{S}$ for all $i < n$ (for all i).*
(v) $\mathrm{H}^i(x_1, \dots, x_r; M)$ *is in $\mathcal{S}$ for all $i < n$ (for all i).*
(vi) $\mathrm{H}^i_\mathfrak{a}(N, M)$ *is in $\mathcal{S}$ for each finite R-module N and for all $i < n$ (for all i).*

When M is finite these conditions are also equivalent to:

(vii) *There is a sequence of length n in $\mathfrak{a}$ that is $\mathcal{S}$-regular on M* (*the same thing for all n*).

Proof. We prove by induction on n that the conditions (i) to (vi) are equivalent for an arbitrary R-module M. First let $n = 1$. Since by hypothesis $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$, conditions (i), (ii) and (v) are equivalent. $\Gamma_{\mathfrak{a}}(N, M) = \Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(N, M)) \cong \mathrm{Hom}_R(N, \Gamma_{\mathfrak{a}}(M))$, when N is finite. If in addition $\mathrm{Supp}_R(M) \subset \mathrm{V}(\mathfrak{a})$, then $\Gamma_{\mathfrak{a}}(N, M) = \mathrm{Hom}_R(N, M)$. Hence (i) $\Rightarrow$ (vi) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Let L be a finite module with $\mathrm{Supp}_R(L) \subset \mathrm{V}(\mathfrak{a})$ and N a finite module as in (iv). By Gruson's theorem [21, Theorem 4.1], there is a finite filtration

$$L = L_0 \supset L_1 \supset \cdots \supset L_s = 0$$

of submodules of L such that each quotient L_{i-1}/L_i is a homomorphic image of a finite number of copies of N. By the exactness of

$$0 \to \mathrm{Hom}_R(L_{i-1}/L_i, M) \to \mathrm{Hom}_R(L_{i-1}, M) \to \mathrm{Hom}_R(L_i, M),$$

we may assume that $s = 1$. If $N^l \to L \to 0$ is exact then $0 \to \mathrm{Hom}_R(L, M) \to \mathrm{Hom}_R(N, M)^l$ is exact. Hence (iv) implies (iii). We have shown that the conditions (i) to (vi) are equivalent when $n = 1$.

Let now $n > 1$ and assume that the conditions are equivalent for all R-modules when n is replaced by $n - 1$. We first reduce to the case $\Gamma_{\mathfrak{a}}(M) = 0$. We already know that $\Gamma_{\mathfrak{a}}(M)$ belongs to $\mathcal{S}$. Hence $\mathrm{Ext}^i_R(N, \Gamma_{\mathfrak{a}}(M))$ belongs to $\mathcal{S}$ for every i and every finite R-module N. It follows that a certain one of the conditions is satisfied by M if and only if the same condition is satisfied by $\bar{M} = M/\Gamma_{\mathfrak{a}}(M)$. We have also used that if N is a finite module, then for every R-module M, $\mathrm{H}^i_{\mathfrak{a}}(N, \Gamma_{\mathfrak{a}}(M)) \cong \mathrm{Ext}^i_R(N, \Gamma_{\mathfrak{a}}(M))$ for all i. Hence we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Let E be an injective hull of M. Then also $\Gamma_{\mathfrak{a}}(E) = 0$. E is a direct sum of injective modules of the form $\mathrm{E}(R/\mathfrak{p})$, where $\mathfrak{a} \not\subset \mathfrak{p}$. Take $s \in \mathfrak{a} \setminus \mathfrak{p}$, then s acts as an automorphism on the module $\mathrm{H}^i(x_1, \ldots, x_r; \mathrm{E}(R/\mathfrak{p}))$ for each i. But these modules are annihilated by $\mathfrak{a}$, they must therefore be zero. Therefore we have for all $i \geqslant 0$:

$$\mathrm{H}^i_{\mathfrak{a}}(E) = 0, \qquad \mathrm{Ext}^i_R(R/\mathfrak{a}, E) = 0, \qquad \mathrm{H}^i_{\mathfrak{a}}(N, E) = 0, \qquad \mathrm{H}^i(x_1, \ldots, x_r; E) = 0.$$

Put $Q = E/M$ and consider the exact sequence $0 \to M \to E \to Q \to 0$. We obtain isomorphisms:

$$\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^{i-1}_{\mathfrak{a}}(Q),$$
$$\mathrm{H}^i(x_1, \ldots, x_n; M) \cong \mathrm{H}^{i-1}(x_1, \ldots, x_n; Q)$$

and

$$\mathrm{Ext}^i_R(N, M) \cong \mathrm{Ext}^{i-1}_R(N, Q)$$

for all finite R-modules N with $\mathrm{Supp}_R(N) \subset \mathrm{V}(\mathfrak{a})$. Also

$$\mathrm{H}^i_{\mathfrak{a}}(N, M) \cong \mathrm{H}^{i-1}_{\mathfrak{a}}(N, Q)$$

for all finite R-modules N. Use the induction hypothesis applied to Q, and conclude that the conditions (i) to (vi) are equivalent.

Let now M be finite.

We show by induction that (vii) is equivalent to the other conditions. Let first $n = 1$. If there is $y \in \mathfrak{a}$ such that $0 :_M y$ is in $\mathcal{S}$, then $\Gamma_{\mathfrak{a}}(M)$ is in $\mathcal{S}$ by 2.3.

Assume that $\Gamma_{\mathfrak{a}}(M)$ is in $\mathcal{S}$. Take $y \in \mathfrak{a}$ such that $y \notin \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_R(M/\Gamma_{\mathfrak{a}}(M))$. The equality $\Gamma_{\mathfrak{a}}(M) = \Gamma_{yR}(M)$ follows from the exact sequence

$$0 \to \Gamma_{yR}\big(\Gamma_{\mathfrak{a}}(M)\big) \to \Gamma_{yR}(M) \to \Gamma_{yR}\big(M/\Gamma_{\mathfrak{a}}(M)\big),$$

observing that $\Gamma_{yR}(\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}}(M)$ and $\Gamma_{yR}(M/\Gamma_{\mathfrak{a}}(M)) = 0$. Hence $0 :_M y$ is in $\mathcal{S}$, i.e. y is $\mathcal{S}$-regular on M.

Let $n > 1$ and assume that the conditions are equivalent for all finite R-modules when n is replaced by $n - 1$. Suppose that $y_1, \dots, y_n$ is an $\mathcal{S}$-sequence on M in $\mathfrak{a}$. Since $0 :_M y_1 \in \mathcal{S}$, $\mathrm{H}^i_{\mathfrak{a}}(0 :_M y_1)$ is in $\mathcal{S}$ for all i, e.g. by (v). Since $y_2, \dots, y_n$ is an $\mathcal{S}$-sequence on M/y_1M, by induction $\mathrm{H}^i_{\mathfrak{a}}(M/y_1M)$ is in $\mathcal{S}$ for all $i < n - 1$. Consider the map $f = y_1 1_M$. Since $\mathrm{H}^i_{\mathfrak{a}}(\mathrm{Ker}\, f)$ and $\mathrm{H}^{i-1}_{\mathfrak{a}}(\mathrm{Coker}\, f)$ are in $\mathcal{S}$ for all $i < n$, by [18, Lemma 3.1], $\mathrm{Ker}\,\mathrm{H}^i_{\mathfrak{a}}(f)$ is in $\mathcal{S}$ for all $i < n$, i.e. $0 :_{\mathrm{H}^i_{\mathfrak{a}}(M)} y_1$ is in $\mathcal{S}$. Lemma 2.3 implies that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is in $\mathcal{S}$ for all $i < n$.

Suppose that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is in $\mathcal{S}$ for all $i < n$. Take y_1 as in the proof of the case $n = 1$. Then y_1 is a non-zerodivisor on $M/\Gamma_{\mathfrak{a}}(M)$ and we may replace M with this module by 2.7(b) and assume that y_1 is a non-zerodivisor on M. We then have a long exact sequence

$$\cdots \to \mathrm{H}^{i-1}_{\mathfrak{a}}(M/y_1M) \to \mathrm{H}^i_{\mathfrak{a}}(M) \xrightarrow{y_1} \mathrm{H}^i_{\mathfrak{a}}(M) \to \cdots.$$

Hence $\mathrm{H}^i_{\mathfrak{a}}(M/y_1M)$ is in $\mathcal{S}$ for all $i < n - 1$. By induction there is a sequence $y_2, \dots, y_n$ in $\mathfrak{a}$ which is an $\mathcal{S}$-sequence on M/y_1M. It follows that $y_1, \dots, y_n$ is $\mathcal{S}$-regular on M by 2.7(a). □

Remark 2.10. Theorem 2.9 can be applied to each Serre subcategory $\mathcal{S}$ mentioned in 2.4, resulting in each case in a number of equivalent conditions.

Part (a) of the following corollary has been proved in [7] by Cuong and Hoang for finite modules but we prove it for arbitrary modules.

Corollary 2.11.

(a) $$\bigcup_{i<n} \mathrm{Supp}_R\big(\mathrm{H}^i_{\mathfrak{a}}(M)\big) = \bigcup_{i<n} \mathrm{Supp}_R\big(\mathrm{Ext}^i_R(R/\mathfrak{a}, M)\big)$$

for every R-module M and every positive integer n.

(b) *If M is a finite R-module, then $\bigcup_{i<n} \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is a closed set for each positive integer n, in particular $\bigcup_i \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is closed.*

Proof. (a)

$$\begin{aligned}\mathfrak{p} \notin \bigcup_{i<n} \mathrm{Supp}_R\big(\mathrm{H}^i_{\mathfrak{a}}(M)\big) \quad &\Leftrightarrow \quad \mathrm{H}^i_{\mathfrak{a}}(M)_{\mathfrak{p}} = 0 \quad \forall i < n,\\ &\Leftrightarrow \quad \mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0 \quad \forall i < n,\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad \operatorname{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}, M_\mathfrak{p}) = 0 \quad \forall i < n,\\
&\Leftrightarrow \quad \operatorname{Ext}^i_R(R/\mathfrak{a}, M)_\mathfrak{p} = 0 \quad \forall i < n,\\
&\Leftrightarrow \quad \mathfrak{p} \notin \bigcup_{i<n} \operatorname{Supp}_R\bigl(\operatorname{Ext}^i_R(R/\mathfrak{a}, M)\bigr).
\end{aligned}$$

(b) Note that the support of a finite R-module is a closed set. □

In [4, Lemma 3.2], part (a) of the following corollary has been proved for finite modules. Now we will prove it in the general case. Moreover the finiteness of $\operatorname{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M))$ under the assumption in (c) has been proved in [14, Theorem B(β)] and [2, Theorem 2.3] for finite modules and in [10, Corollary 2.7] for *weakly Laskerian* modules. A module M is weakly Laskerian when each quotient M/N has just finitely many associated primes. Here we prove, for an arbitrary module M under the assumption in (c), that the finiteness of $\operatorname{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M))$ is equivalent to the finiteness of $\operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M))$.

Corollary 2.12. *Let M be an R-module and let n be a positive integer. Put*

$$P_n = \bigcup_{i=0}^{n-1} \operatorname{Supp}_R\bigl(\operatorname{Ext}^i_R(R/\mathfrak{a}, M)\bigr).$$

Then

(a) $\operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M)) \cup P_n = \operatorname{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M)) \cup P_n$.
(b) $\operatorname{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M)) \subset \operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M)) \cup P_n$.
(c) *If $\mathrm{H}^i_\mathfrak{a}(M)$ has finite support for all $i < n$, then $\operatorname{Ass}_R(\mathrm{H}^n_\mathfrak{a}(M))$ is a finite set if and only if $\operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M))$ is a finite set.*

Proof. (a) If $\mathfrak{p} \notin \bigcup_{i<n} \operatorname{Supp}_R(\mathrm{H}^i_\mathfrak{a}(M))$, then by [1, Corollary 2.3]

$$\operatorname{Ass}_R\bigl(\operatorname{Ext}^n_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}, M_\mathfrak{p})\bigr) = \operatorname{Ass}_R\bigl(\mathrm{H}^n_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p})\bigr).$$

Now it is clear that, $\mathfrak{p}$ is not in the left side if and only if it is not in the right side by Corollary 2.11(a).

(b) follows from (a).

(c) Use part (a) and Corollary 2.11(a). □

The following theorem is a generalization of [1, Theorem 2.5] to Serre subcategories of the category of R-modules that satisfy the condition $C_\mathfrak{a}$.

Theorem 2.13. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules such that $\mathcal{S}$ satisfies the condition $C_\mathfrak{a}$. If $\operatorname{Ext}^{t-j}_R(R/\mathfrak{a}, \mathrm{H}^j_\mathfrak{a}(M))$ is in $\mathcal{S}$ for $t = n, n+1$ and for all $j < n$, then $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is in $\mathcal{S}$ if and only if $\mathrm{H}^n_\mathfrak{a}(M)$ is in $\mathcal{S}$.*

Proof. Use [1, Proposition 2.1(c)] and the condition $C_\mathfrak{a}$. □

Lemma 2.14. *Let M be a finite module and let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$.*

(a) *$M/\mathfrak{a}M$ is in $\mathcal{S}$, if and only if there are sequences in $\mathfrak{a}$ of arbitrary length which are $\mathcal{S}$-regular on M.*
(b) *If $M/\mathfrak{a}M$ is not in $\mathcal{S}$, then every sequence in $\mathfrak{a}$ which is $\mathcal{S}$-regular on M, can be extended to a maximal one. All maximal $\mathcal{S}$-sequences on M in $\mathfrak{a}$, have the same length. This length n is the smallest n such that $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is not in $\mathcal{S}$.*

Proof. (a) The modules $\mathrm{H}^i(x_1, \dots, x_r; M)$ are finite modules with support in $\operatorname{Supp}_R(M/\mathfrak{a}M)$, where $\mathfrak{a} = (x_1, \dots, x_r)$. Now the following fact is used; Let $\mathcal{S}$ be a Serre subcategory and let X and Y be finite R-modules with $\operatorname{Supp}_R(X) \supset \operatorname{Supp}_R(Y)$. If X is in $\mathcal{S}$, then so is Y. (Confer 2.17 below.) Hence if $M/\mathfrak{a}M$ is in $\mathcal{S}$, then $\mathrm{H}^i(x_1, \dots, x_r; M)$ belong to $\mathcal{S}$ for all i. Then by 2.9 (v) $\Leftrightarrow$ (vii) there are sequences in $\mathfrak{a}$ of arbitrary length which are $\mathcal{S}$-regular on M.

Conversely, if there are $\mathcal{S}$-sequences on M in $\mathfrak{a}$ of arbitrary length, then by 2.9 (v) $\Leftrightarrow$ (vii), the modules $\mathrm{H}^i(x_1, \dots, x_r; M)$ are in $\mathcal{S}$ for all i. In particular $\mathrm{H}^r(x_1, \dots, x_r; M)$ which is isomorphic to $M/\mathfrak{a}M$ is in $\mathcal{S}$.

Hence if $M/\mathfrak{a}M$ is not in $\mathcal{S}$, then there is a bound on the length of sequences in $\mathfrak{a}$, which are $\mathcal{S}$-regular on M and any such sequence can be extended to a maximal one.

(b) Let $y_1, \dots, y_n$ be a maximal sequence in $\mathfrak{a}$, which is $\mathcal{S}$-regular on M. If we show that $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is not in $\mathcal{S}$ then it follows that there is no longer one by 2.9(ii) $\Leftrightarrow$ (vii). We do this by induction on n. First we reduce to the case that y_1 is a non-zerodivisor on M.

Since M is finite and $0 :_M y_1$ is in $\mathcal{S}$, by Lemma 2.3, $\Gamma_{y_1R}(M)$, which is equal to $0 :_M {y_1}^l$ for some integer l, is in $\mathcal{S}$. Thus $y_1, \dots, y_n$ is an $\mathcal{S}$-sequence on $\bar{M} = M/\Gamma_{y_1R}(M)$ by 2.7(b) and y_1 is a regular element on $\bar{M}$. From the exact sequence $0 \to \Gamma_{y_1R}(M) \to M \to \bar{M} \to 0$, we get the exact sequence

$$\operatorname{Ext}^n_R\big(R/\mathfrak{a}, \Gamma_{y_1R}(M)\big) \to \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^n_R(R/\mathfrak{a}, \bar{M}) \to \operatorname{Ext}^{n+1}_R\big(R/\mathfrak{a}, \Gamma_{y_1R}(M)\big).$$

The two outer terms belong to $\mathcal{S}$. Hence $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is in $\mathcal{S}$ if and only if $\operatorname{Ext}^n_R(R/\mathfrak{a}, \bar{M})$ is in $\mathcal{S}$. Now we may assume that y_1 is a non-zerodivisor on M. The exact sequence $0 \to M \xrightarrow{y_1} M \to M/y_1M \to 0$ yields the exact sequence

$$\operatorname{Ext}^{n-1}_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^{n-1}_R(R/\mathfrak{a}, M/y_1M) \to \operatorname{Ext}^n_R(R/\mathfrak{a}, M).$$

By definition, $y_2, \dots, y_n$ is a maximal $\mathcal{S}$-regular sequence on M/y_1M. By the induction hypothesis $\operatorname{Ext}^{n-1}_R(R/\mathfrak{a}, M/y_1M)$ is not in $\mathcal{S}$, thus $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is not in $\mathcal{S}$. $\square$

Definition 2.15. Let M be a finite module and let $\mathfrak{a}$ be an ideal of R such that $M/\mathfrak{a}M$ is not in $\mathcal{S}$, where $\mathcal{S}$ is a Serre subcategory of the category of R-modules, that satisfies the condition $C_{\mathfrak{a}}$. We denote the common length of all maximal $\mathcal{S}$-regular sequences on M in $\mathfrak{a}$ by $\mathcal{S}\text{-}\operatorname{depth}_{\mathfrak{a}}(M)$.

Example 2.16. Let M be a finite module. The following are some examples of $\mathcal{S}\text{-}\operatorname{depth}_{\mathfrak{a}}(M)$ with $\mathcal{S}$ as in Example 2.4.

(a) It is the same as ordinary $\operatorname{depth}_{\mathfrak{a}}(M)$.
(b) It is the same as f-$\operatorname{depth}_{\mathfrak{a}}(M)$ (filter-depth) which has been defined in the local case in [17] and [15, Definition 3.3].
(c) It is the same as g-$\operatorname{depth}_{\mathfrak{a}}(M)$ (generalized depth) which has been defined in the local case in [19, Definition 4.2].

Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. Let Z be the set of all prime ideals $\mathfrak{p}$ such that $R/\mathfrak{p}$ belongs to $\mathcal{S}$. This subset of $\operatorname{Spec}(R)$ is closed under specialization, because there is an exact sequence $R/\mathfrak{p} \to R/\mathfrak{q} \to 0$ whenever $\mathfrak{p} \subset \mathfrak{q}$. The Serre subcategory defined by Z (see Example 2.4(e)) is in general larger than $\mathcal{S}$ but these two Serre subcategories have the same finite modules as the following lemma shows.

Lemma 2.17. *For a finite R-module M the following are equivalent*:

(i) *M belongs to $\mathcal{S}$.*
(ii) $\operatorname{Supp}_R(M) \subset Z$.
(iii) $\operatorname{Ass}_R(M) \subset Z$.
(iv) $\operatorname{Min}\operatorname{Ass}_R(M) \subset Z$.

Proof. (i) ⇒ (iii) For each $\mathfrak{p} \in \operatorname{Ass}_R(M)$ there is an exact sequence $0 \to R/\mathfrak{p} \to M$.

(iii) ⇒ (iv) Trivial.

(iv) ⇒ (ii) For each $\mathfrak{q} \in \operatorname{Supp}_R(M)$, there is $\mathfrak{p} \in \operatorname{Min}\operatorname{Ass}_R(M)$ such that $\mathfrak{q} \supset \mathfrak{p}$ and Z is closed under specialization.

(ii) ⇒ (i) There is a finite filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$$

such that $M_{i-1}/M_i \cong R/\mathfrak{p}_i$ where $\mathfrak{p}_i \in \operatorname{Supp}_R(M)$ for $i = 1, \ldots, t$. □

Theorem 2.18. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, such that $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$. Let M be a finite module and let $\mathfrak{a} = (x_1, \ldots, x_r)$ be an ideal of R, such that $M/\mathfrak{a}M$ is not in $\mathcal{S}$. Then*

(a) $\mathcal{S}$-$\operatorname{depth}_{\mathfrak{a}}(M) = \min\{i \mid \mathrm{H}^i_{\mathfrak{a}}(M)$ *is not in* $\mathcal{S}\}$.
(b) $\mathcal{S}$-$\operatorname{depth}_{\mathfrak{a}}(M) = \min\{i \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is not in* $\mathcal{S}\}$.
(c) $\mathcal{S}$-$\operatorname{depth}_{\mathfrak{a}}(M) = \min\{i \mid \mathrm{H}^i(x_1, \ldots, x_r; M)$ *is not in* $\mathcal{S}\}$.
(d) $\mathcal{S}$-$\operatorname{depth}_{\mathfrak{a}}(M) = \min\{\operatorname{depth}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ *and* $R/\mathfrak{p}$ *is not in* $\mathcal{S}\}$.
(e) *If $\mathcal{S}$ closed under taking injective hulls then* $\mathcal{S}$-$\operatorname{depth}_{\mathfrak{a}}(M) = \min\{\mathcal{S}$-$\operatorname{depth}_{\mathfrak{p}}(M) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ *and* $R/\mathfrak{p}$ *is not in* $\mathcal{S}\}$.

Proof. (a), (b) and (c) follow from 2.9 and 2.14.

(d) Let $n = \mathcal{S}$-$\operatorname{depth}_{\mathfrak{a}}(M)$ and consider an arbitrary prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ such that $R/\mathfrak{p}$ is not in $\mathcal{S}$. Since, by (b), the finite R-module $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is in $\mathcal{S}$ when $i < n$, we get from 2.17 that $\mathfrak{p} \notin \operatorname{Supp}_R(\operatorname{Ext}^i_R(R/\mathfrak{a}, M))$ for $i < n$. Hence $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for all $i < n$, and therefore $\operatorname{depth}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geqslant n$. On the other hand, by (b), the finite R-module $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is not in $\mathcal{S}$. Hence by 2.17 there is $\mathfrak{q} \in \operatorname{Supp}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M))$ such that $R/\mathfrak{q}$ is not in $\mathcal{S}$. Since $\operatorname{Ext}^n_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}}, M_{\mathfrak{q}}) \neq 0$ we conclude that $\operatorname{depth}_{\mathfrak{a}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) = n$.

(e) Let $n = \mathcal{S}\text{-}\operatorname{depth}_{\mathfrak{a}}(M)$ and let $\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ such that $R/\mathfrak{p}$ is not in $\mathcal{S}$. From the definition we get the inequality $\mathcal{S}\text{-}\operatorname{depth}_{\mathfrak{p}}(M) \geqslant \mathcal{S}\text{-}\operatorname{depth}_{\mathfrak{a}}(M)$. By (d) there is $\mathfrak{q} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ such that $R/\mathfrak{q}$ is not in $\mathcal{S}$ and $\operatorname{depth}_{\mathfrak{a}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) = n$. Therefore there is a prime ideal $\mathfrak{p}$ with $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{q}$ such that $\operatorname{depth} M_{\mathfrak{p}} = n$ [6, Proposition 1.2.10]. Then $R/\mathfrak{p}$ is not in $\mathcal{S}$ and $\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$. It follows from (d) that $\mathcal{S}\text{-}\operatorname{depth}_{\mathfrak{p}}(M) \leqslant \operatorname{depth}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = n$. Consequently the prime ideal $\mathfrak{p}$ satisfies $\mathcal{S}\text{-}\operatorname{depth}_{\mathfrak{p}}(M) = n$. $\square$

3. Study of local cohomology modules from above

Theorems 3.1 and 3.3 are the main results of this section. They show that the study of local cohomology from above depends just on the support.

Theorem 3.1. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules and let M be a finite R-module and n a non-negative integer. Then the following are equivalent*:

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is in $\mathcal{S}$ for all $i > n$.*
(ii) $\mathrm{H}^i_{\mathfrak{a}}(N)$ *is in $\mathcal{S}$ for all $i > n$ and for every finite R-module N such that* $\operatorname{Supp}_R(N) \subset \operatorname{Supp}_R(M)$.
(iii) $\mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p})$ *is in $\mathcal{S}$ for all* $\mathfrak{p} \in \operatorname{Supp}_R(M)$ *and all $i > n$.*
(iv) $\mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p})$ *is in $\mathcal{S}$ for all* $\mathfrak{p} \in \operatorname{Min}\operatorname{Ass}_R(M)$ *and all $i > n$.*

Proof. We use descending induction on n. So we may assume that all conditions are equivalent when n is replaced by $n+1$.

(i) $\Rightarrow$ (iii). We want to show that $\mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{p})$ is in $\mathcal{S}$ for each $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Suppose the contrary and let $\mathfrak{p} \in \operatorname{Supp}_R(M)$ be maximal of those $\mathfrak{p} \in \operatorname{Supp}_R(M)$ such that $\mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{p})$ is not in $\mathcal{S}$. Since $\mathfrak{p} \in \operatorname{Supp}_R(M)$, there is by [3, Chap. (ii), §4, n^o 4, Proposition 20] a non-zero map $f : M \to R/\mathfrak{p}$. Let $\mathfrak{b} \supsetneqq \mathfrak{p}$ be the ideal of R such that $\operatorname{Im} f = \mathfrak{b}/\mathfrak{p}$. The exact sequence $0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0$, yields the exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(M) \to \mathrm{H}^{n+1}_{\mathfrak{a}}(\operatorname{Im} f) \to \mathrm{H}^{n+2}_{\mathfrak{a}}(\operatorname{Ker} f).$$

Since $\operatorname{Supp}_R(\operatorname{Ker} f) \subset \operatorname{Supp}_R(M)$, by induction $\mathrm{H}^{n+2}_{\mathfrak{a}}(\operatorname{Ker} f)$ belongs to $\mathcal{S}$. It follows that $\mathrm{H}^{n+1}_{\mathfrak{a}}(\operatorname{Im} f)$ belongs to $\mathcal{S}$. There is a filtration

$$0 = N_t \subset N_{t-1} \subset N_{t-2} \subset \cdots \subset N_0 = R/\mathfrak{b}$$

of submodules of $R/\mathfrak{b}$, such that for each $0 \leqslant i \leqslant t$, $N_{i-1}/N_i \cong R/\mathfrak{q}_i$ where $\mathfrak{q}_i \in \mathrm{V}(\mathfrak{b})$. Then by the maximality of $\mathfrak{p}$, $\mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{q}_i)$ is in $\mathcal{S}$. Use the exact sequences $0 \to N_i \to N_{i-1} \to R/\mathfrak{q}_i \to 0$, to conclude that $\mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{b})$ is in $\mathcal{S}$. Next the exact sequence $0 \to \operatorname{Im} f \to R/\mathfrak{p} \to R/\mathfrak{b} \to 0$, yields the exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(\operatorname{Im} f) \to \mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{p}) \to \mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{b}).$$

It follows that $\mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{p})$ is in $\mathcal{S}$ which is a contradiction.

(iii) $\Rightarrow$ (ii). Use a filtration for N as above.

(iv) $\Rightarrow$ (iii). Let $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $\mathfrak{p} \supset \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Min}\operatorname{Ass}_R(M)$. Hence $\mathfrak{p} \in \operatorname{Supp}_R(R/\mathfrak{q})$. Applying (i) $\Rightarrow$ (iii), it follows that $\mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p})$ is in $\mathcal{S}$ for all $i > n$. $\square$

Corollary 3.2. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules and n a non-negative integer. If M and N are finite R-modules such that* $\operatorname{Supp}_R(N) = \operatorname{Supp}_R(M)$*, then* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is in $\mathcal{S}$ for all $i > n$ if and only if* $\mathrm{H}^i_{\mathfrak{a}}(N)$ *is in $\mathcal{S}$ for all $i > n$.*

When $\mathcal{S}$ is no longer an arbitrary Serre subcategory of the category of R-modules but satisfies the condition $C_{\mathfrak{a}}$, we are able to weaken the condition (iii) in 3.1 to require that $\mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p})$ is in $\mathcal{S}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$, just for $i = n+1$.

Theorem 3.3. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules satisfying the condition $C_{\mathfrak{a}}$ and n a non-negative integer. Then for each finite R-module M the conditions in Theorem* 3.1 *are equivalent to*:

(v) $\mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{p})$ *is in $\mathcal{S}$ for all* $\mathfrak{p} \in \operatorname{Supp}_R(M)$.

Proof. (v) $\Rightarrow$ (iv). We prove by induction on $i \geqslant n+2$ that $\mathrm{H}^i_{\mathfrak{a}}(R/\mathfrak{p})$ is in $\mathcal{S}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$. It is enough to treat the case $i = n+2$. Suppose that $\mathrm{H}^{n+2}_{\mathfrak{a}}(R/\mathfrak{p})$ is not in $\mathcal{S}$ for some $\mathfrak{p} \in \operatorname{Supp}_R(M)$. It follows that $\mathfrak{a} \not\subset \mathfrak{p}$, since otherwise $\mathrm{H}^{n+2}_{\mathfrak{a}}(R/\mathfrak{p}) = 0$, because $n+2 > 0$. Take $x \in \mathfrak{a} \setminus \mathfrak{p}$ and put $N = R/(\mathfrak{p} + xR)$. Then $\operatorname{Supp}_R(N) \subset \operatorname{Supp}_R(M)$. We have a finite filtration

$$0 = N_t \subset N_{t-1} \subset N_{t-2} \subset \cdots \subset N_0 = N$$

such that $N_{i-1}/N_i \cong R/\mathfrak{p}_i$ for each $1 \leqslant i \leqslant t$ where $\mathfrak{p}_i \in \operatorname{Supp}_R(M)$. Using the exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(N_i) \to \mathrm{H}^{n+1}_{\mathfrak{a}}(N_{i-1}) \to \mathrm{H}^{n+1}_{\mathfrak{a}}(R/\mathfrak{p}_i)$$

for each $1 \leqslant i \leqslant t$, shows that $\mathrm{H}^{n+1}_{\mathfrak{a}}(N)$ is in $\mathcal{S}$. Consider the exact sequence $0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to N \to 0$, which induces the following exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(N) \to \mathrm{H}^{n+2}_{\mathfrak{a}}(R/\mathfrak{p}) \xrightarrow{x} \mathrm{H}^{n+2}_{\mathfrak{a}}(R/\mathfrak{p}).$$

This shows that $0 :_{\mathrm{H}^{n+2}_{\mathfrak{a}}(R/\mathfrak{p})} x$ is in $\mathcal{S}$. Since $\mathrm{H}^{n+2}_{\mathfrak{a}}(R/\mathfrak{p})$ is $\mathfrak{a}$-torsion, by 2.3 $\mathrm{H}^{n+2}_{\mathfrak{a}}(R/\mathfrak{p})$ is in $\mathcal{S}$, which is a contradiction. $\square$

Remark 3.4. In Theorems 3.1 and 3.3 we may specialize $\mathcal{S}$ to any of the Serre subcategories given in 2.4, to obtain characterizations of artinianness, vanishing, finiteness of the support, etc. See also [13, Proposition 2.3].

Dibaei and Yassemi gave some parts of them in [9, Theorem 3.2 and 3.9] for the case of artinianness. In the case of vanishing, i.e., when $\mathcal{S}$ merely consists of zero modules, some parts of them were studied in [11, Theorem 2.2] and [8, Theorem 2.1]. These authors used Gruson's theorem, [21, Theorem 4.1], while we just used the maximal condition in a noetherian ring.

Definition 3.5. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ be an ideal of R and M an R-module. We define

$$\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M) = \min\bigl\{n \geqslant 0 \bigm| \mathrm{H}^i_{\mathfrak{a}}(M) \text{ is in } \mathcal{S} \text{ for all } i > n\bigr\}.$$

For example when $\mathcal{S}=\{0\}$, then $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)=\mathrm{cd}(\mathfrak{a}, M)$ and when $\mathcal{S}$ is the class of artinian modules, then $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)=\mathrm{q}(\mathfrak{a}, M)$ as in [13] and [9].

In the following we study the main properties of this invariant.

Proposition 3.6. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. Let $\mathfrak{a}$ be an ideal of R. The following statements hold.*

(a) *Let $\mathcal{S}_1, \mathcal{S}_2$ be two Serre subcategories of the category of R-modules such that $\mathcal{S}_1 \subset \mathcal{S}_2$. Then $\mathrm{t}_{\mathcal{S}_2}(\mathfrak{a}, M) \leqslant \mathrm{t}_{\mathcal{S}_1}(\mathfrak{a}, M)$ for every finite R-module M. In particular $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M) \leqslant \mathrm{cd}(\mathfrak{a}, M)$ for each Serre subcategory $\mathcal{S}$ of the category of R-modules.*
(b) *If M and N are finite R-modules s.t. $\mathrm{Supp}_R(N) \subset \mathrm{Supp}_R(M)$, then $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, N) \leqslant \mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)$ and equality holds if $\mathrm{Supp}_R(N)=\mathrm{Supp}_R(M)$.*
(c) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite R-modules. Then $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)=\max\{\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M'), \mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M'')\}$.*
(d) $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, R)=\sup\{\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, N) \mid N$ *is a finite R-module*$\}$.
(e) $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)=\sup\{\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R(M)\}$.
(f) $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)=\sup\{\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Min}\,\mathrm{Ass}_R(M)\}$.

If $\mathcal{S}$ satisfies the condition $C_{\mathfrak{a}}$, then the following statements hold:

(g) $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)=\min\{r \geqslant 0 \mid \mathrm{H}_{\mathfrak{a}}^{r+1}(R/\mathfrak{p}) \in \mathcal{S}$ *for all* $\mathfrak{p} \in \mathrm{Supp}_R(M)\}$.
(h) *For each integer i with $1 \leqslant i \leqslant \mathrm{t}_{\mathcal{S}}(\mathfrak{a}, M)$, there exists $\mathfrak{p} \in \mathrm{Supp}_R(M)$ with $\mathrm{H}_{\mathfrak{a}}^{i}(R/\mathfrak{p})$ not in $\mathcal{S}$.*
(i) $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, R)=\min\{r \geqslant 0 \mid \mathrm{H}_{\mathfrak{a}}^{r+1}(R/\mathfrak{p}) \in \mathcal{S}$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)\}$.
(j) $\mathrm{t}_{\mathcal{S}}(\mathfrak{a}, R)=\min\{r \geqslant 0 \mid \mathrm{H}_{\mathfrak{a}}^{r+1}(N) \in \mathcal{S}$ *for all finite R-modules N*$\}$.

Proof. (a) By definition.

(b) follows from 3.1.

(c) The inequality "$\geqslant$," holds by (b) and we get the opposite inequality from the following exact sequence

$$\cdots \to \mathrm{H}_{\mathfrak{a}}^{i}(M') \to \mathrm{H}_{\mathfrak{a}}^{i}(M) \to \mathrm{H}_{\mathfrak{a}}^{i}(M'') \to \cdots.$$

The assertions (d), (e) and (f) follow from Theorem 3.1(i) $\Leftrightarrow$ (ii), (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iv), respectively.

(g), (h), (i) and (j) follow from 3.3. $\square$

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